

Available online at www.sciencedirect.com

J. Math. Anal. Appl. 323 (2006) 1300–1310

Journal of
 MATHEMATICAL
 ANALYSIS AND
 APPLICATIONS

www.elsevier.com/locate/jmaa

On the near differentiability property of Banach spaces

Patrick N. Dowling^{a,*}, Mangatiana A. Robdera^b

^a Department of Mathematics and Statistics, Miami University, Oxford, OH 45056, USA

^b Lot VK 28 bis Ambohimano, Antananarivo, Madagascar

Received 12 May 2005

Available online 22 December 2005

Submitted by Steven G. Krantz

Abstract

Let μ be a scalar measure of bounded variation on a compact metrizable abelian group G . Suppose that μ has the property that for any measure σ whose Fourier–Stieltjes transform $\hat{\sigma}$ vanishes at ∞ , the measure $\mu * \sigma$ has Radon–Nikodým derivative with respect to λ , the Haar measure on G . Then L. Pigno and S. Saeki showed that μ itself has Radon–Nikodým derivative. Such property is not shared by vector measures in general. We say that a Banach space X has the near differentiability property if every X -valued measure of bounded variation shares the above property. We prove that Banach spaces with the Radon–Nikodým property have the near differentiability property, while Banach spaces with the near differentiability property enjoy the near Radon–Nikodým property. We also show that the Banach spaces $L^1[0, 1]$ and L^1/H_0^1 have the near differentiability property. Lastly, we show that Banach spaces with the near differentiability property have type II- Λ -Radon–Nikodým property, whenever Λ is a Riesz subset of type 0 of \hat{G} .

© 2005 Elsevier Inc. All rights reserved.

Keywords: Radon–Nikodým property; Riesz set of type 0

1. Introduction

The Banach space $L^1[0, 1]$ plays a central role in the theory of the Radon–Nikodým property in Banach spaces. Over the past couple of decades much work has been done on the study of properties that are weaker than the Radon–Nikodým property and enjoyed by $L^1[0, 1]$. Two important examples of such properties are the analytic Radon–Nikodým property of Bukhvalov

* Corresponding author.

E-mail addresses: dowlinpn@muohio.edu (P.N. Dowling), robdera@yahoo.com (M.A. Robdera).

and Danilevich [3], and the near Radon–Nikodým property of Kaufman, Petrakis, Riddle and Uhl [9]. The analytic Radon–Nikodým property can be generalized to the setting of compact metrizable abelian groups and the so-called types I- and II- Λ -Radon–Nikodým properties can be defined, where Λ is a subset of the dual group of a compact metrizable abelian group (see Section 5 for details). In particular, $L^1[0, 1]$ has type I- Λ -Radon–Nikodým property if and only if Λ is a Riesz set [6]. On the other hand, the near Radon–Nikodým property of a Banach space X is a property that is defined in terms of the representability of a class of bounded operators from $L^1[0, 1]$ into X .

The property we introduce in this paper, the near differentiability property, has the flavor of both the near Radon–Nikodým property and type I- Λ -Radon–Nikodým property. In fact we will prove that Banach spaces with the near differentiability property have both the near Radon–Nikodým property and type I- Λ -Radon–Nikodým property, when Λ is a suitably small set (a Riesz subset of type 0). However, the principal motivation for studying the near differentiability property is that the near differentiability property is the vector-valued version of a property of the scalars that was proved by Pigno and Saeki [10,11] (see also [15, p. 147]). In particular, we will see that $L^1[0, 1]$ and the quotient space L^1/H_0^1 enjoy the near differentiability property. Furthermore, the near differentiability property is quite a stable Banach space property; it is separably determined, it is an isomorphic invariant, it is a three-space property, and it lifts to some tensor products (see Section 4 for the details).

2. Preliminaries

Throughout this paper, G is a compact metrizable abelian group and \widehat{G} its discrete dual group. We denote by λ the normalized Haar measure on G ; $\widehat{(\cdot)}$, the Fourier–Stieltjes transformation. The σ -field of Borel subsets of G will be denoted by $\mathcal{B}(G)$. For a Banach space X , $\mathcal{M}^1(G, X)$ (respectively $\mathcal{M}^\infty(G, X)$) will denote the Banach space of X -valued countably additive measures on G that are of bounded variation (respectively of bounded average range). $L^1(G, X)$ (respectively $L^\infty(G, X)$), denotes the usual Banach space of (equivalence classes of) λ -Bochner integrable (respectively essentially bounded) functions on G with values in the Banach space X . If X is the scalar field, then $\mathcal{M}^1(G, X)$, $L^1(G, X)$ and $L^\infty(G, X)$, will be denoted by $\mathcal{M}^1(G)$, $L^1(G)$ and $L^\infty(G)$, respectively. The subspace of $\mathcal{M}^1(G)$ consisting of scalar measures whose Fourier–Stieltjes transforms vanish at ∞ will be denoted by $\mathcal{M}_0(G)$. A measure μ in $\mathcal{M}^1(G, X)$ is said to be (Bochner) *differentiable* if there exists a function $g \in L^1(G, X)$ such that $\mu(A) = \int_A g \, d\lambda$ for every measurable subset A of G . In what follows, we always identify the set of differentiable measures with $L^1(G, X)$.

A Banach space X is said to have the *Radon–Nikodým property*, if every X -valued λ -continuous countably additive measure of bounded variation on G is differentiable. Although originally defined from the point of view of measure theoretic method, it is a well-known fact that the Radon–Nikodým property can be naturally characterized by properties of X -valued operators on $L^1(G)$. Indeed, a bounded linear operator $T : L^1(G) \rightarrow X$ is said to be (Bochner) *representable* if there exists $g \in L^\infty(G, X)$ such that $Tf = \int_G fg \, d\lambda$ for every $f \in L^1(G)$. It is then well known that a Banach space X has the Radon–Nikodým property if and only if every bounded linear operator $T : L^1(G) \rightarrow X$ is representable [4].

A property closely related to the Radon–Nikodým property is the so called near Radon–Nikodým property. A bounded linear operator $T : L^1(G) \rightarrow X$ is said to be *nearly representable* if for every completely continuous operator $D : L^1(G) \rightarrow L^1(G)$ the composition $T \circ D$ is representable. A Banach space X is said to have the *near Radon–Nikodým property*, if every nearly

representable operator $T: L^1(G) \rightarrow X$ is representable. This notion was first introduced and studied by R. Kaufman, M. Petrakis, L. Riddle, and J.J. Uhl Jr. [9]. It is apparent that this notion was approached from the point of view of operator theoretic method. The first part of this paper is devoted to the investigation of the possible vector measure theoretic analogue of the notion of near Radon–Nikodým property for Banach spaces and its ramifications to the study of subsets of discrete abelian groups. The motivation for our approach is the result of L. Pigno and S. Saeki [10,11] announced in the abstract which pointed out a rather remarkable property of scalar measures.

For a vector measure μ in $\mathcal{M}^1(G, X)$ and a scalar measure σ in $\mathcal{M}^1(G)$, we define the convolution $\mu * \sigma$ on $\mathcal{B}(G)$ by

$$\mu * \sigma(A) = \int_G \int_G \chi_A(x+y) d\sigma(y) d\mu(x).$$

It is an exercise to show that $\mu * \sigma \in \mathcal{M}^1(G, X)$ and that $\|\mu * \sigma\|_1 \leq \|\mu\|_1 \|\sigma\|_1$, where $\|\cdot\|_1$ denote the variation norms in $\mathcal{M}^1(G, X)$ and $\mathcal{M}^1(G)$. The following definition can be considered as the measure theoretic counterpart of the notion of nearly representable operators.

Definition 1. Let G be a compact metrizable abelian group. Let μ be in $\mathcal{M}^1(G, X)$. Then μ is said to be *nearly differentiable* if for every scalar measure $\sigma \in \mathcal{M}_0(G)$ the convolution $\mu * \sigma$ is differentiable.

With this terminology, the result of L. Pigno and S. Saeki [10,11] can be restated as follows:

Theorem 2 (L. Pigno and S. Saeki). *Every nearly differentiable scalar measure on a compact metrizable abelian group G is differentiable.*

Remark 1. The original theorem of L. Pigno and S. Saeki was actually proved for the more general case of non-discrete locally compact abelian group G . However, for our purposes, we only consider compact metrizable abelian groups in the sequel.

Remark 2. It should be noted that in Definition 1 we did not assume that the measure μ in $\mathcal{M}^1(G, X)$ is absolutely continuous with respect to normalized Haar measure λ on G . This is an unnecessary assumption because, for each $x^* \in X^*$, the scalar measure $x^* \circ \mu$ is nearly differentiable and so by Theorem 2, $x^* \circ \mu$ is differentiable. Consequently, μ is absolutely continuous with respect to λ .

3. The near differentiability property

Simple natural adjustments to the proof show that the property of scalar measures described in the above Theorem 2 extends to measures with values in a finite dimensional Banach space. In general, however, we will quickly see that this property is not shared by the vector measures. That is to say, there are indeed infinite dimensional Banach spaces X on which one can find X -valued nearly differentiable non-differentiable measures. Such observation allows us to define a new class of Banach spaces. Before introducing such class, we need to recall some well-known and classical facts concerning vector-valued measures on G and vector-valued operators on $L^1(G)$.

Suppose X is a Banach space, and let $T: L^1(G) \rightarrow X$ be a bounded linear operator. Then T defines an X -valued λ -continuous measure μ_T by $\mu_T(A) = T(\chi_A)$, for every measurable subset

A of G , where χ_A denotes the characteristic function of the set A . It is well known that the operator T is representable if and only if the associated measure μ_T is differentiable [4, p. 62].

On the other hand, every $\mu \in \mathcal{M}^1(G, X)$ defines an operator $T_\mu: L^1(G) \rightarrow L^1(G, X)$ by $T_\mu f = \mu * f$, for every $f \in L^1(G)$. If the measure μ is λ -continuous then μ is differentiable if and only if the associated convolution operator T_μ is representable. For a scalar measure σ , the convolution operator T_σ associated to a measure σ is completely continuous if and only if $\sigma \in \mathcal{M}_0(G)$ (see, for example, [4]). For connection on convolution operators with respect to vector measures, we refer the reader to [14].

In the sequel we will make use of the following technical theorem of Kalton [8] about random measures associated to operators from L^1 into $L^1(\mu, X)$ where X is a Banach space.

Theorem 3. *Suppose that*

- (1) K is a compact metric space and λ is a Radon measure on K ;
- (2) Ω is a Polish space and μ is a finite Radon measure on Ω ;
- (3) X is a separable Banach space;
- (4) $T: L^1(\lambda) \rightarrow L^1(\mu, X)$ is a bounded operator.

Then there is a map $t \mapsto \mu_t$ ($\Omega \rightarrow \mathcal{M}^1(K, X)$) such that for every A Borel subset of K , the map $t \mapsto \mu_t(A)$ is Borel measurable from $\Omega \rightarrow X$ and

- (a) *for each A Borel subset of K , $\int_\Omega |\mu_t|(A) d\mu(t) \leq \|T\| \lambda(A)$;*
- (b) *if $f \in L^1(\lambda)$, then for μ -a.e. t , $f \in L^1(|\mu_t|)$;*
- (c) *$Tf(t) = \int_\Omega f d\mu_t$, μ -a.e. for every $f \in L^1(\lambda)$.*

We refer to this theorem as the Kalton's representation theorem. Finally, N. Randrianantoanina and E. Saab [12] showed the following useful result:

Theorem 4. *Under the assumptions of Theorem 3 the following are equivalent:*

- (1) *the operator T is representable;*
- (2) *for μ -a.e. t , μ_t has Bochner integrable density with respect to λ .*

We are now ready to introduce our definition.

Definition 5. A Banach space X is said to have the *near differentiability property* if for every compact metrizable abelian group G , every nearly differentiable measure $\mu \in \mathcal{M}^1(G, X)$ is differentiable.

Remark 3. If G is a compact metrizable abelian group, then $\mathcal{B}(G)$ is countably generated. Therefore, the range of any $\mu \in \mathcal{M}^1(G, X)$ is contained in a separable subspace of X . Consequently, the near differentiability property of X is determined by the separable subspaces of X ; that is, if every separable subspace of X has near differentiability property, then X has the near differentiability property.

As mentioned above in Theorem 2, the scalar field is an example of Banach space with the near differentiability property. Our first example of infinite dimensional Banach space with the

near differentiability property is given by $L^1(G)$. The idea of the proof of this fact is contained in the proof of the fact that L^1 has the near Radon–Nikodým property [9].

Proposition 6. *Let G be a compact metrizable abelian group. Then the Banach space $L^1(G)$ has the near differentiability property.*

Proof. Let μ be an element of $\mathcal{M}^1(G, L^1(G))$ and suppose that μ is not differentiable. Then, by a similar argument as in A. Costé's Theorem [4], (see also [14]), the convolution operator $T_\mu : L^1(G) \rightarrow L^1(G, L^1(G))$ associated to μ is not representable. The Banach space $L^1(G, L^1(G))$ can be identified to $L^1(G \times G)$. It is known that $L^1(G \times G)$ has the near Radon–Nikodým property [9]. Repeating sine qua non the argument in the proof of the fact that L^1 has the near Radon–Nikodým property in [9], one can construct a convolution operator T_σ where $\sigma \in \mathcal{M}_0(G)$ such that $T_\mu T_\sigma$ is not representable. A simple computation shows that $T_\mu T_\sigma = T_{\mu * \sigma}$. We can therefore conclude that the measure μ is not nearly differentiable and so the proof is complete. \square

Let us observe that any differentiable measure is clearly nearly differentiable. Consequently, any Banach space with the Radon–Nikodým property has the near differentiability property. Since it is well known that $L^1(G)$ fails the Radon–Nikodým property, Proposition 6 shows that the near differentiability property is strictly a weaker property than the Radon–Nikodým property. On the other hand, the following theorem suggests itself.

Theorem 7. *Every Banach space with the near differentiability property has the near Radon–Nikodým property.*

Proof. Suppose X is a Banach space with the near differentiability property. Let $T : L^1(G) \rightarrow X$ be a nearly representable operator. We want to show that T is representable. As mentioned in the preliminaries, it suffices to show that the naturally associated measure μ_T is differentiable. Thus the proof is complete if we show that μ_T is nearly differentiable. To this end, fix $\sigma \in \mathcal{M}_0(G)$. For every measurable subset A of G , we have

$$\mu_T * \sigma(A) = \int_G \int_G \chi_A(x+y) d\sigma(y) d\mu_T(x) = TT_{\bar{\sigma}}\chi_A, \quad (*)$$

where $\bar{\sigma}$ is the scalar measure defined by $\bar{\sigma}(A) = \sigma(-A)$ for every $A \in B(G)$. Since clearly $\hat{\bar{\sigma}}$ vanishes at infinity, the convolution operator $T_{\bar{\sigma}}$ is completely continuous and therefore our hypothesis on the operator T ensures that $TT_{\bar{\sigma}}$ is representable. It follows that the measure $\mu_T * \sigma$ is differentiable. This completes the proof. \square

As a corollary we see that c_0 (and consequently any Banach space containing isomorphic copy of c_0), since it fails the near Radon–Nikodým property, is an example of a Banach space failing the near differentiability property. At this moment, we do not know if the converse of Theorem 7 holds.

Theorem 8. *The Banach space L^1/H_0^1 has the near differentiability property.*

Proof. Let $\mu : \mathcal{B}(G) \rightarrow L^1/H_0^1$ be a nearly differentiable measure of bounded variation. First note that μ is absolutely continuous with respect to (normalized) Haar measure λ on G . Applying

the Hahn decomposition theorem, we get a pairwise disjoint sequence of elements (E_n) , of $\mathcal{B}(G)$, such that $\bigcup_{n=1}^{\infty} E_n = G$ and

$$(n-1)\lambda(A \cap E_n) \leq |\mu|(A \cap E_n) \leq n\lambda(A \cap E_n) \quad \text{for all } A \in \mathcal{B}(G).$$

For each $n \geq 1$, define $\mu_n : \mathcal{B}(G) \rightarrow L^1/H_0^1$ by

$$\mu_n(A) = \mu(A \cap E_n) \quad \text{for all } A \in \mathcal{B}(G).$$

The measure μ_n is of bounded average range, so μ_n corresponds to a bounded linear operator $T_n : L^1 \rightarrow L^1/H_0^1$ (i.e., for all $B \in \mathcal{B}(G)$, define $T_n(\chi_B) = \mu_n(B)$ and extend T_n linearly). By a result of Ghoussoub and Rosenthal [7, Proposition V.2., p. 330], there exists a bounded linear operator $R_n : L^1 \rightarrow L^1$ with $\|R_n\| \leq 2\|T_n\|$ and $q(R_n) = T_n$, where q is the natural quotient map from L^1 to L^1/H_0^1 . For each $n \geq 1$, define measures $\nu_n : \mathcal{B}(G) \rightarrow L^1$ by $\nu_n(B) = R_n(\chi_B)$, for all $B \in \mathcal{B}(G)$. Now define ν on $\mathcal{B}(G)$ by

$$\nu(B) = \sum_{n=1}^{\infty} \nu_n(B) \quad \text{for all } B \in \mathcal{B}(G).$$

We claim that ν is an L^1 -valued measure of bounded variation with $q(\nu(B)) = \mu(B)$ for all $B \in \mathcal{B}(G)$. Firstly,

$$|\nu|(G) \leq \sum_{n=1}^{\infty} |\nu_n|(G) \leq \sum_{n=1}^{\infty} 2|\mu_n|(G) = 2 \sum_{n=1}^{\infty} |\mu_n|(E_n) = 2|\mu|\left(\bigcup_{n=1}^{\infty} E_n\right) = 2|\mu|(G) < \infty.$$

Secondly, for $B \in \mathcal{B}(G)$ we have

$$q(\nu(B)) = q\left(\sum_{n=1}^{\infty} \nu_n(B)\right) = \sum_{n=1}^{\infty} q(\nu_n(B)) = \sum_{n=1}^{\infty} \mu_n(B) = \sum_{n=1}^{\infty} \mu(B \cap E_n) = \mu(B).$$

This proves the claim.

Suppose that $\sigma \in M_0(G)$. It is easily seen that $q(\nu * \sigma) = q(\nu) * \sigma = \mu * \sigma$. Since μ is a nearly differentiable measure, $\mu * \sigma$ is differentiable and consequently, $q(\nu * \sigma)$ is differentiable. By an argument very similar to the proof of a theorem of Edgar (see [4, p. 211]), one sees that the measure $\nu * \sigma$ is differentiable. This means that ν is a nearly differentiable measure. Therefore, since L^1 has the near differentiability property, the measure ν is differentiable. It follows that the measure $\mu = q(\nu)$ is also differentiable. This completes the proof. \square

Remark 4. It should be noted that Theorem 8 can be extended to other settings. The ingredients used in the proof of Theorem 8 are that H_0^1 has the Radon–Nikodým property, H_0^1 is a weak*-closed subspace of the space of measures on the unit circle \mathbb{T} , and $L^1(\mathbb{T})$ has the near differentiability property. In particular, if R is a reflexive subspace of L^1 , then L^1/R has the near differentiability property. Furthermore, an easy modification of the proof of Theorem 8 yields the following result, the proof of which we omit.

Corollary 9. *Let X be a dual Banach space with the near differentiability property and let Y be a weak*-closed subspace of X which also has the Radon–Nikodým property. Then the quotient space, X/Y , has the near differentiability property.*

4. Stability and three-space properties

In this section, we attempt to investigate other possible classes of Banach spaces which share the near differentiability property. First we recall that a Banach space X is said to be *semi-embed* in a Banach space Y if there exists a bounded one-to-one operator $S: X \rightarrow Y$ such that the image of the closed unit ball of X is closed in Y . In [1], J. Bourgain and H.P. Rosenthal showed that if a separable Banach space X semi-embeds in a Banach space with the Radon–Nikodým property, then X itself has the Radon–Nikodým property (actually, they attribute this result to F. Delbaen). This proof of this result has been modified by Randrianantoanina and Saab [12, Proposition 3]. This modification of Randrianantoanina and Saab can be readily seen to give us the following result.

Theorem 10. *Let Y be a Banach space with near differentiability property and let X be a separable Banach space that semi-embeds in Y . Then X has the near differentiability property.*

Let \mathcal{S} denote the *Ghoussoub–Rosenthal class* of Banach spaces [7], that is, the class of separable Banach spaces closed under semi-embedding and containing L^1 . In light of the above result, one sees that the class of Banach spaces with the near differentiability property is large enough to contain those Banach spaces whose separable subspaces are in \mathcal{S} .

Another stability problem for Banach spaces is to ask whether a certain property passes from a Banach space X to the Bochner integrable function space $L^1([0, 1], X)$. While we would expect that the near differentiability property lifts from X to the function space $L^1([0, 1], X)$, we can only prove the following special case.

Theorem 11. *The Banach space $L^1([0, 1], X)$ has the near differentiability property whenever the Banach space X has the Radon–Nikodým property.*

Proof. Suppose X has the Radon–Nikodým property. To show that $L^1([0, 1], X)$ has the near differentiability property, it suffices to show that each of its separable subspaces have the near differentiability property. Consequently, we may assume that X is separable.

Let G be a compact metrizable abelian group and let μ be an $L^1([0, 1], X)$ -valued nearly differentiable measure on $\mathcal{B}(G)$. We want to show that μ is a differentiable measure. To do this, it suffices to show the associated convolution operator $T_\mu: L^1(G) \rightarrow L^1(G, L^1([0, 1], X))$, defined by $T_\mu(f) = \mu * f$, is representable [13, Theorem 71]. For each $x^* \in X^*$ consider the $L^1[0, 1]$ -valued measure μ^{x^*} defined by $\mu^{x^*}(A)(t) = \langle \mu(A)(t), x^* \rangle$ for every measurable subset A of G . We claim that μ^{x^*} is differentiable. Simple computation shows that for each $\sigma \in \mathcal{M}_0(G)$ one has $\mu^{x^*} * \sigma = (x^* \mu) * \sigma = x^*(\mu * \sigma)$. It follows that the measure μ^{x^*} is nearly differentiable. Since $L^1[0, 1]$ has the near differentiability property (Proposition 6), we then see that the measure μ^{x^*} is in fact differentiable as claimed.

On the other hand, since the Banach space $L^1(G, L^1([0, 1], X))$ is isometrically isomorphic to $L^1(G \times [0, 1], X)$, we may consider the convolution operator T_μ as an operator into $L^1(G \times [0, 1], X)$. Let $t \mapsto \nu_t (G \times [0, 1] \rightarrow \mathcal{M}^1(G, X))$ be the random measure given by the Kalton representation theorem (Theorem 3). Let (x_n^*) be a countable dense subset of X^* . For each $n \in \mathbb{N}$, the map $t \mapsto x_n^* \circ \nu_t (G \times [0, 1] \rightarrow \mathcal{M}^1(G))$ is Borel measurable and defines an operator $T^{x_n^*}: L^1(G) \rightarrow L^1(G \times [0, 1])$ by

$$(T^{x_n^*} f)(t) = \int_G f(\omega) d(x_n^* \circ \nu_t)(\omega), \quad \text{for all } f \in L^1(G).$$

It is easy to see that $T^{x_n^*} = T_{\mu^{x_n^*}}$ where $T_{\mu^{x_n^*}}$ is the convolution operator associated to $\mu^{x_n^*}$. It then follows from our previous claim that $T^{x_n^*}$ is representable and therefore for a.e. $t \in G \times [0, 1]$, we get by Theorem 4 that $x_n^* v_t$ has a Bochner integrable derivative with respect to λ , and in particular, $x_n^* v_t$ is absolutely continuous with respect to λ for a.e. $t \in G \times [0, 1]$. Consequently, v_t is absolutely continuous with respect to λ for a.e. $t \in G \times [0, 1]$. Now, since X has the Radon–Nikodým property v_t is differentiable with respect to λ for a.e. $t \in G \times [0, 1]$. An appeal to Theorem 4 now finishes the proof. \square

Of course, the obvious question arising from Theorem 11 is: does $L^1([0, 1], X)$ have the near differentiability property whenever X has the near differentiability property?

Let X be a Banach space. A Schauder decomposition of X is a sequence $(X_n)_{n=1}^\infty$ of non-trivial closed subspaces of X such that every $x \in X$ can be expressed uniquely in the form $x = \sum_{n=1}^\infty x_n$, where $x_n \in X_n$ for every $n \in \mathbb{N}$. A Schauder decomposition $(X_n)_{n=1}^\infty$ is boundedly complete if, whenever $(\sum_{n=1}^m x_n)_{m=1}^\infty$ is a bounded sequence with $x_n \in X_n$ for every $n \in \mathbb{N}$, then $\sum_{n=1}^\infty x_n$ converges.

The proof of the following theorem follows directly from [2, Theorem 4].

Theorem 12. *Let X be a Banach space having a boundedly complete Schauder decomposition $(X_n)_{n=1}^\infty$. Then X has near differentiability property if each X_n , $n \in \mathbb{N}$, has the near differentiability property.*

An immediate consequence of this theorem that follows from the techniques developed in [2] is the following stability result for projective tensor products of Banach spaces.

Corollary 13. *Let U be a Banach space with an unconditional basis and X a Banach space. Then $U \hat{\otimes} X$, the projective tensor product of U and X , has the near differentiability property if and only if U and X have the near differentiability property.*

In [9], the authors show that the near Radon–Nikodým property is a “three-space property.” We also show that the near differentiability property is a three-space property. Moreover, our proof can easily be seen to provide yet another proof of the fact that the Radon–Nikodým property is a three-space property.

Theorem 14. *Let X be a Banach space and let Y a closed subspace of X . Suppose that both Y and X/Y have the near differentiability property. Then X has the near differentiability property.*

Proof. Let G be a compact abelian metrizable group. Let μ be an X -valued measure on $\mathcal{B}(G)$ which is nearly differentiable. Define an operator $T : C(G) \rightarrow X$ by

$$T(f) = \int_G f d\mu, \quad \text{for all } f \in C(G).$$

Let $\sigma \in M_0(G)$ and define $T_\sigma : C(G) \rightarrow C(G)$ by $T_\sigma(f) = \sigma * f$, for all $f \in C(G)$. Note that the operator TT_σ from $C(G)$ to X can be written as

$$TT_\sigma(f) = \int_G f d(\mu * \bar{\sigma}), \quad \text{for all } f \in C(G).$$

Since $\sigma \in M_0(G)$, $\bar{\sigma} \in M_0(G)$, and hence the measure $\mu * \bar{\sigma}$ is differentiable, since μ is nearly differentiable. Therefore, by [4, the proof of Theorem 4, p. 173], TT_σ is nuclear. Let $q: X \rightarrow X/Y$ be the natural quotient mapping. Note that the operator qT is nuclear because its representing measure is $q(\mu)$, and $q(\mu)$ is differentiable because it is near differentiable and X/Y has the near differentiability property. Hence there are operators $R: C(G) \rightarrow \ell^\infty$, $\lambda: \ell^\infty \rightarrow \ell^1$ and $S: \ell^1 \rightarrow X/Y$, such that λ is nuclear and $qT = S\lambda R$ [4, p. 170]. Since q is onto, by the lifting property of ℓ^1 , there exists a bounded linear operator $\tilde{S}: \ell^1 \rightarrow X$ such that $S = q\tilde{S}$. Consequently, we have the following commutative diagram:

$$\begin{array}{ccccc} C(G) & \xrightarrow{T_\sigma} & C(G) & \xrightarrow{T} & X & \xrightarrow{q} & X/Y \\ & & \downarrow R & & \nwarrow \tilde{S} & & \uparrow S \\ & & \ell^\infty & \xrightarrow{\lambda} & \ell^1 & & \end{array}$$

Note that $q(T - \tilde{S}\lambda R) = qT - q\tilde{S}\lambda R = qT - S\lambda R = 0$. Thus the operator $U = T - \tilde{S}\lambda R$ maps $C(G)$ into Y . For each $\sigma \in M_0(G)$, the operator $UT_\sigma = TT_\sigma - \tilde{S}\lambda RT_\sigma$ is nuclear, since TT_σ is nuclear and $\tilde{S}\lambda RT_\sigma$ is nuclear because λ is nuclear. If ν is the representing measure for U , then $\nu * \bar{\sigma}$ is the representing measure for UT_σ , for each $\sigma \in M_0(G)$. Hence $\nu * \bar{\sigma}$ is differentiable for each $\sigma \in M_0(G)$, and since Y has the near differentiability property, ν is differentiable. This in turn means that U is nuclear. Thus the operator $T = U + \tilde{S}\lambda R$ is nuclear, so its representing measure, μ , is differentiable. This completes the proof. \square

5. Near differentiability property and Radon–Nikodým types

We first recall two other properties of Banach spaces closely related to the Radon–Nikodým property: the so-called Radon–Nikodým property types. A Banach space X is said to have type *I- Λ -Radon–Nikodým property* (respectively *II- Λ -Radon–Nikodým property*) [5] if every $\mu \in \mathcal{M}^\infty(G, X)$ (respectively $\mu \in \mathcal{M}^1(G, X)$ and $\mu \ll \lambda$) satisfying $\hat{\mu}(\gamma) = 0$ for $\gamma \notin \Lambda$ is differentiable.

Recall that a subset Λ of \hat{G} is said to be a *Riesz set of type 0* if for every measure $\sigma \in \mathcal{M}_0(G)$, there exists a function $\varphi \in L^1(G)$ such that $\hat{\sigma}(\gamma) = \hat{\varphi}(\gamma)$ for every $\gamma \in \Lambda$; that is, $\mathcal{M}_0(G)|_\Lambda = L^1(G)|_\Lambda$. It is easily seen that Sidon sets are Riesz sets of type 0, and that Riesz sets of type 0 are Riesz sets [11].

Theorem 15. *Let G be a compact metrizable abelian group. Then every Banach space with the near differentiability property has type II- Λ -Radon–Nikodým property, whenever Λ is a Riesz subset of type 0 of \hat{G} .*

Proof. Let $\mu \in \mathcal{M}^1(G, X)$ be a λ -continuous X -valued measure satisfying $\hat{\mu}(\gamma) = 0$ for $\gamma \notin \Lambda$. We want to show that μ is differentiable. Since X has near differentiability property, the proof will be complete if we show that μ is nearly differentiable. To see this, consider $\sigma \in \mathcal{M}_0(G)$. Since Λ is a Riesz set of type 0, there exists $\varphi \in L^1(G)$ such that $\hat{\sigma}(\gamma) = \hat{\varphi}(\gamma)$ for every $\gamma \in \Lambda$. Thus it follows that

$$(\widehat{\mu * \sigma})(\gamma) = \hat{\mu}(\gamma)\hat{\sigma}(\gamma) = \hat{\mu}(\gamma)\hat{\varphi}(\gamma) = (\widehat{\mu * \varphi})(\gamma)$$

for $\gamma \in \hat{G}$, and hence $\mu * \sigma = \mu * \varphi \in L^1(G, X)$; that is, μ is nearly differentiable. This completes the proof. \square

Remark 5. Note that the proof of Theorem 15 actually shows that Riesz sets of type 0 are Riesz sets. The way to see this is to note that since $L^1[0, 1]$ has the near differentiability property, $L^1[0, 1]$ has type II- Λ -Radon–Nikodým property, whenever Λ is a Riesz subset of type 0 of \widehat{G} . A result of Edgar [6] now shows that Λ is a Riesz set.

For every $\Lambda \subset \widehat{G}$, if we denote

$$\mathcal{M}_\Lambda^1(G, X) = \{\mu \in \mathcal{M}^1(G, X): \hat{\mu}(\gamma) = 0 \text{ for } \gamma \notin \Lambda\},$$

it is apparent that the above proof essentially contains the following fact.

Proposition 16. *Let G be a compact metrizable abelian group. Let $\Lambda \subset \widehat{G}$ be a Riesz set of type 0. For any Banach space X , every X -valued Λ -measure is nearly differentiable; that is,*

$$\mathcal{M}_\Lambda^1(G, X) * \mathcal{M}_0(G) \subset L_\Lambda^1(G, X).$$

Our next result may be regarded as of independent interest.

Theorem 17. *Let G be a compact metrizable abelian group. Let X be a Banach space with the near differentiability property. Let $\Lambda \subset \widehat{G}$ be a Riesz set of type 0 and let $\mu \in \mathcal{M}^1(G, X)$ be a λ -continuous measure. Then*

$$\mu * \mathcal{M}_{\widehat{G} \setminus \Lambda}^1(G) \subset L^1(G, X) \quad \Rightarrow \quad \mu \in L^1(G, X).$$

Proof. Let $\mu \in \mathcal{M}^1(G, X)$ be λ -continuous and such that $\mu * \mathcal{M}_{\widehat{G} \setminus \Lambda}^1(G) \subset L^1(G, X)$. Since X has the near differentiability property, we are done if we show that μ is nearly differentiable. To see this, let $\sigma \in \mathcal{M}_0(G)$. Our assumption on Λ alerts us that there exists an integrable function $\varphi \in L^1(G)$ such that $\hat{\sigma}(\gamma) = \hat{\varphi}(\gamma)$ for $\gamma \in \Lambda$. It follows that the measure $\sigma - \varphi \cdot \lambda \in \mathcal{M}_{\widehat{G} \setminus \Lambda}^1(G)$. Therefore by our assumption, there exists $f \in L^1(G, X)$ such that $\mu * (\sigma - \varphi \cdot \lambda) = f \cdot \lambda$. Thus $\mu * \sigma = \mu * \varphi + f \cdot \lambda \in L^1(G, X)$ and hence, since X has the near differentiability property, $\mu \in L^1(G, X)$. This completes the proof. \square

Straightforward adjustment of the above proof yields the following property of scalar measures.

Corollary 18. *Let G be a compact metrizable abelian group. A measure $\mu \in \mathcal{M}^1(G)$ is λ -absolutely continuous whenever the measure $\mu * \sigma$ is λ -absolutely continuous, for every $\sigma \in \mathcal{M}_{\widehat{G} \setminus \Lambda}^1(G)$, where Λ is a Riesz subset of type 0 of \widehat{G} .*

Let us notice that for $\Lambda = \widehat{G}$, the type II- Λ -Radon–Nikodým property coincides with the usual Radon–Nikodým property. Thus the result of Proposition 17 also suggests the following:

Corollary 19. *Let G be a compact metrizable abelian group. Then a Banach space X with the near differentiability property has the Radon–Nikodým property whenever it has type II- $(G \setminus \Lambda)$ -Radon–Nikodým property, for some Λ Riesz subset of type 0 of \widehat{G} .*

Proof. Let $\mu \in \mathcal{M}^1(G, X)$ be a λ -continuous measure. Then one has $\mu * \mathcal{M}_{\widehat{G} \setminus \Lambda}^1(G) \subset \mathcal{M}_{\widehat{G} \setminus \Lambda}^1(G, X)$. Since X has type II- $(G \setminus \Lambda)$ -Radon–Nikodým property, $\mathcal{M}_{\widehat{G} \setminus \Lambda}^1(G, X) \subset$

$L^1(G, X)$, the result of Theorem 17 applies and implies that $\mu \in L^1(G, X)$. This completes the proof. \square

References

- [1] J. Bourgain, H.P. Rosenthal, Applications of the theory of semi-embeddings to Banach space theory, *J. Funct. Anal.* 52 (1983) 149–188.
- [2] Q. Bu, J. Diestel, P. Dowling, E. Oja, Types of Radon–Nikodým properties for the projective tensor product of Banach spaces, *Illinois J. Math.* 47 (4) (2003) 1303–1326.
- [3] A.V. Bukhvalov, A.A. Danilevich, Boundary properties of analytic and harmonic functions with values in Banach spaces, *Mat. Zametki* 31 (2) (1982) 203–214.
- [4] J. Diestel, J.J. Uhl Jr., *Vector Measures*, Math. Surveys, vol. 15, Amer. Math. Soc., Providence, RI, 1977.
- [5] P.N. Dowling, Radon–Nikodým properties associated with subsets of countable discrete abelian groups, *Trans. Amer. Math. Soc.* 327 (2) (1991) 879–890.
- [6] G.A. Edgar, Banach spaces with the analytic Radon–Nikodým property and compact abelian groups, in: *Almost Everywhere Convergence*, Columbus, OH, 1988, Academic Press, Boston, MA, 1989, pp. 195–213.
- [7] N. Ghoussoub, H.P. Rosenthal, Martingales, G_δ -embeddings and quotients of L^1 , *Math. Ann.* 264 (1983) 321–332.
- [8] N.J. Kalton, Isomorphisms between L_p -function spaces when $p < 1$, *J. Funct. Anal.* 42 (3) (1981) 299–337.
- [9] R. Kaufman, M. Petrakis, L. Riddle, J.J. Uhl Jr., Nearly representable operators, *Trans. Amer. Math. Soc.* 312 (1989) 315–333.
- [10] L. Pigno, S. Saeki, Measures whose transforms vanish at infinity, *Bull. Amer. Math. Soc.* 79 (1973) 800–801.
- [11] L. Pigno, S. Saeki, Fourier–Stieltjes transforms which vanish at infinity, *Math. Z.* 141 (1975) 83–91.
- [12] N. Randrianantoanina, E. Saab, Stability of some types of Radon–Nikodým properties, *Illinois J. Math.* 39 (3) (1995) 416–430.
- [13] M. Robdera, On the analytic complete continuity property of Banach spaces and convolution operators, PhD dissertation, University of Missouri–Columbia, 1996.
- [14] M.A. Robdera, P. Saab, Convolution operators associated with vector measures, *Glasg. Math. J.* 40 (1998) 367–384.
- [15] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge Stud. Adv. Math., vol. 25, Cambridge Univ. Press, 1991.